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ORIGINAL PAPER

Modeling of nonlinear boundary value problems in enzyme-catalyzed reaction diffusion processes

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Abstract A mathematical model of steady state mono-layer potentiometric biosensor is developed. The model is based on non stationary diffusion equations containing a non linear term related to Michaelis-Menten kinetics of the enzymatic reaction. This paper presents a complex numerical method (He's variational iteration method) to solve the non-linear differential equations that describe the diffusion coupled with a Michaelis-Menten kinetics law. Approximate analytical expressions for substrate concentration and corresponding current response have been derived for all values of saturation parameter α and reaction diffusion parameter K using variational iteration method. These results are compared with available limiting case results and are found to be in good agreement. The obtained results are valid for the whole solution domain.

Keywords Non-linear · Boundary value problems · Enzyme-catalyzed reaction · Michaelis-meten kinetics

1 Introduction

Biosensors are analytical devices that combine the selectivity and specificity of a biologically active compound with a signal transducer and an electronic amplifier [1–3].

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The transducer coverts the physico-chemical change of the biological sensing elements usually an enzyme, resulting from the interaction with analyte into an output concentration dependent signal. The biosensors are classified according to the nature of the physical transducer. The amperometric biosensors measure the changes of the current on a working indicator electrode due to direct oxidation of the products of the biochemical reaction [2–4]. Potentiometric biosensor have assumed great importance in both theoretical and applied work. In this case the potential at the electrode is held constant while the current flow is measured. Biosensors combine the execute selectivity of biology with the processing power of modern microelectronics and optoelectronics to offer powerful new analytical tools with major applications in medicine, environment diagnosis and the food processing industries. Biosensors can be mass produced at low cost. Biosensors yield a signal which is proportional to the concentration of a measured analyte.

Modelling of biosensors is of a crucial importance to understand their behavior. Normally it is not possible to measure the concentration of subtracts inside the enzyme membranes with analytical devices. Hence mathematical models in biosensors have been developed and used as an important tool to study the analytical characteristics of actual biosensors. Biosensor modeling have began with the work of Goldman and co-workers [1]. They have published an extensive mathematical treatment of substrate and product distribution in membranes containing enzymes. Sundaram et al. [2] derived the equations describing the kinetics of reaction in an enzyme membrane immersed in a substrate solution. Kasche and co-workers [3] presented a model and equations describing steady-state catalysis by an enzyme immobilized in a spherical gel particles, and showed that catalysis by an bounded enzyme at low substrate concentrations differs obviously from catalysis by an unbound enzyme. Blaedel et al. [4] derived equations for steady state fluxes of substrate and product through a membrane in simple systems.

Some of the equations are solved for the fluxes, for the substrate (or) product concentrations. Gough et al. [5] have simulated the performance of a cylindrical biosensor for glucose monitoring at steady state. Another mathematical model have been used for the description of steady state and non-steady state behaviour of a multi-membrane multi-enzyme amperometric biosensor. In Jobst et al. [6], a finite difference scheme was used for the discretization of the model equation. Bacha et al. [7,8] have developed a model that take into account a variety of configuration designs to describe the behavior of amperometric biosensor for glucose monitoring. Recently Baronas et al. [9–11,24] have developed a mathematical model to examine the dynamic response of amperometric biosensors in stirred and non-stirred solutions.

Using the implicit finite difference scheme [12], the influence of the substrate concentration as well as maximal enzymatic rate on the biosensor was investigated [13]. The explicit scheme is usually easier to program, however the implicit scheme has a higher simulation speed [12,14]. The developed program was employed also to generate multiple biosensor response data for specific analyzer of various concentrations. The general time-dependent problem has been tackled previously by Carr [15] using Fourier analysis, and the steady state problem has been examined by Brady and Carr [16] via digital simulation using orthogonal collocation methods. Tran-Minh and Broun [17] have also examined both the steady state and transient potentiometric

response using digital simulation methods. Lyons and co-workers [19,23] have developed the approximate analytical expression of substrate concentration and current only for small values of α and K using Laplace transform and 'magic' approximation (Eq. 22). Recently Phanthong and Somasundrum [25] derived the steady state current occurring at a microdisk electrode when a product from an immobilized enzyme reacts at the electrode.

To my knowledge no rigorous analytical solutions for non-linear steady state concentration/current for polymer modified electrodes for all values of α (saturation parameter) and K (reaction-diffusion parameter) have been reported. It should be pointed out that, complete solutions have not yet been obtained even for steady state behaviour because of the non-linearity inherent in Michaelis-Menten kinetics. In this paper, we have derived a new, simple and closed analytical expressions of concentration and current using variational iteration method [20,26].

2 Mathematical formulation of the problem

During an enzyme-catalyzed reaction

$$E + S \overset{K_M}{\longleftrightarrow} ES \overset{k_2}{\longrightarrow} E + P \tag{1}$$

the substrate (S) binds to the enzyme (E) to form enzyme-substrate complex ES. While it is a part of this complex, the substrate is converted to product (P). The rate of the appearance of the product depends on the concentration of the substrate. In the simplest case, the diffusion of substrate molecules is neglected and steady-state conditions are assumed for the enzyme reaction, the mathematical model of enzyme kinetics is given by Michaelis-Menten equation:

$$\nu = \frac{dP}{dT} = -\frac{dS}{dT} = \frac{V_{\max} S}{K_M + S} \tag{2}$$

where ν is the rate of the enzymatic reaction, $V_{\max}$ is the maximal enzymatic rate attainable with that amount of enzyme, when the enzyme is fully saturated with substrate, K_M is the Michaelis constant, S is the substrate concentration, P is concentration of

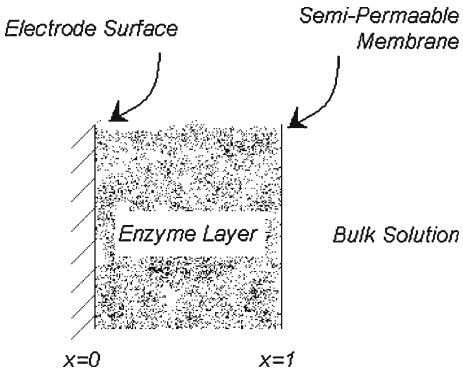

Fig. 1 Schematic representation of a potentiometric enzyme electrode

the product, and T is time.The basic model used in this work and a definition of the coordinate system are shown in Fig. 1. Let us assume the symmetrical geometry of the electrode and homogeneous distribution of immobilized enzyme in the enzyme membrane. Considering one-dimensional diffusion, coupling the reaction (1) with the diffusion described by Fick's law leads to the following non-linear reaction diffusion equations:

$$D_S \frac{\partial^2 S}{\partial \chi^2} - \frac{V_{\max} S}{K_M + S} = 0, \quad 0 < \chi \leq 1 \tag{3a}$$

$$D_P \frac{\partial^2 P}{\partial \chi^2} + \frac{V_{\max} S}{K_M + S} = 0, \quad 0 < \chi \leq 1 \tag{3b}$$

where D_S and D_P are diffusion coefficients of the substrate and product respectively. Now, for a potentiometric biosensor, the boundary conditions are given by [9]

$$S = S_0; \quad P = 0 \quad \text{at } \chi = 1 \tag{4a}$$

$$\frac{\partial S}{\partial \chi} + \frac{\partial P}{\partial \chi} = 0 \quad \text{at } \chi = 0 \tag{4b}$$

We make the above non-linear partial differential equations (Eqs. 3a, 3b) in dimensionless form by defining the following parameters s, p, x, K and α, where $s(= S/ks^{\infty}$ where s^{∞} denotes the bulk concentration of the substrate and k denotes the first order rate constant for substrate reaction at the polymer site), $p(= P/ks^{\infty})$ represents dimensionless concentrations and $x(= \chi/L$ where χ denotes distance and L denotes the thickness of the layer) represents dimensionless distance. Here $\alpha(= ks^{\infty}/K_M$ where K_M denotes Michaelis constant) denotes a saturation parameter and K $(= kL^2/D_S)$ denotes reaction diffusion parameter. We also assume that D_S and D_P are equal. The Eqs. 3a, 3b reduces to the following dimensionless form:

$$\frac{\partial^2 s}{\partial x^2} - \frac{Ks}{1 + \alpha s} = 0, \quad 0 < x \leq 1 \tag{5a}$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{Ks}{1 + \alpha s} = 0, \quad 0 < x \leq 1 \tag{5b}$$

Again, Eqs. 5a, 5b are non-linear partial differential equation. Now the boundary conditions (Eqs. 4a, 4b) becomes

$$s(x = 1) = 1; \quad p(x = 1) = 0 \tag{6a}$$

$$\frac{\partial s}{\partial x} + \frac{\partial p}{\partial x} = 0 \quad \text{at } x = 0 \tag{6b}$$

The concentration of the substrate can be obtained by solving the non-linear Eq. 5a for the above boundary conditions using variation iteration method. Adding the two Eqs. 5a, 5b, we obtain $\partial^2 s/\partial^2 x + \partial^2 p/\partial^2 x = 0$. This expression may be readily integrated twice, using the boundary conditions (6a) and (6b), we obtain the relation

$s(x)+p(x)=1$. Using this result the concentration of product P through the film may be readily evaluated. Recently Rajendran et al. [18], solved a non linear Eq. 5a for the boundary conditions $s(x=0)=f(K)$. The variational iteration method [20,21,26] has been successfully applied to finding the solution of non linear differential equation in closed form. The basic concept of variational iteration method is summarized briefly here for completeness.

3 Basic concepts in the variational iteration method

To illustrate the basic concepts of variational iteration method (VIM), we consider the following non-linear partial differential equation:

$$L\left[s(x)\right]+N\left[s(x)\right]=g(x) \tag{7}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is a given continuous function [20,22,26]. According to the variational iteration method, we can construct a correct functional as follows:

$$s_{n+1}(x)=s_n(x)+\int_0^x \lambda\left[L\left[s_n(\xi)\right]+N\left[\widetilde{s_n}(\xi)\right]-g(\xi)\right]d\xi \tag{8}$$

where λ is a general Lagrange multiplier [20–22] which can be identified optimally via variational theory, s_n is the nth approximate solution, and $\widetilde{s_n}$ denotes a restricted variation, i.e., $\delta\,\widetilde{s_n}=0$.

4 Steady state solution using variational iteration method

The non-linear Eq. 5a is solved for the above boundary conditions using variational iteration method. Using the boundary conditions (Eqs. 6a and 6b), we begin with initial guess satisfying the above boundary conditions

$$s_0=1-a+ax^2 \tag{9}$$

where a ($a<1$) is free parameter. The unknown parameter a is to be determined by using first iteration results. The variational iteration method [20,21] has been successfully applied to finding the solution of differential equation in closed form. Using variation iteration method, we can write the correction functional of Eq. 8 as follows

$$s_{n+1}(x)=s_n(x)+\int_0^x \lambda\left[\left(1+\alpha s_n(\xi)\right)\frac{\partial^2 s_n(\xi)}{\partial\xi^2}-K s_n(\xi)\right]d\xi \tag{10}$$

$$s_{n+1}(x) = s_n(x) + \int_0^x \lambda \left[\frac{\partial^2 s_n(\xi)}{\partial \xi^2} + \overbrace{\alpha s_n(\xi) \frac{\partial^2 s_n(\xi)}{\partial \xi^2}}^{\sim} - \overbrace{K s_n(\xi)}^{\sim} \right] d\xi \tag{11}$$

Taking variation with respect to the independent variable s_n, noticing that $\delta s_n(0) = 0$

$$\delta s_{n+1}(x) = \delta s_n(x) + \delta \int_0^x \lambda \left[\frac{\partial^2 s_n(\xi)}{\partial \xi^2} + \overbrace{\alpha s_n(\xi) \frac{\partial^2 s_n(\xi)}{\partial \xi^2}}^{\sim} - \overbrace{K s_n(\xi)}^{\sim} \right] d\xi \tag{12}$$

For all variational δs_n and $\delta s_n{}'$, implying the following stationary conditions

$$\delta s_n : 1 - \lambda'(\xi)\big|_{\xi=x} = 0 \tag{13a}$$

$$\delta s_n' : \lambda(\xi)|_{\xi=x} = 0 \tag{13b}$$

$$\delta s_n : \lambda''(\xi)\big|_{\xi=x} = 0 \tag{13c}$$

The Lagrange multiplier can be identified as

$$\lambda(\xi) = \xi - x \tag{14}$$

Substituting the Lagrange multiplier $\lambda(\xi)$ in the iteration formula (Eq. 11) we get the following approximation

$$s_{n+1}(x) = s_n(x) + \int_0^x (\xi - x) \left[\frac{\partial^2 s_n(\xi)}{\partial \xi^2} + \alpha s_n(\xi) \frac{\partial^2 s_n(\xi)}{\partial \xi^2} - K s_n(\xi) \right] d\xi \tag{15}$$

Substitution of the value of s_0 from the equation 9 in the above Eq. 15,we get.

$$s_1(x) = 1 - a + \left[\frac{2a^2\alpha - 2a\alpha + K - Ka}{2} \right] x^2 - \left[\frac{2a^2\alpha - Ka}{12} \right] x^4 \tag{16}$$

For simplification we can write the above equation as

$$s_1(x) = A + Bx^2 - Cx^4 \tag{17}$$

where

$$A = 1 - a; \quad B = \left[\frac{2a^2\alpha - 2a\alpha + K - Ka}{2} \right]; \quad C = \left[\frac{2a^2\alpha - Ka}{12} \right] \tag{18}$$

Table 1 The numerical value of a (for Eq. 16) for various value of α and K

	$\alpha = 0.1$	$\alpha = 1$	$\alpha = 10$
$K = 0.01$	0.004530	0.002497	0.000455
$K = 0.1$	0.043937	0.024740	0.004544
$K = 1$	0.335869	0.224235	0.045293
$K = 4$	0.739357	0.637950	0.178966
$K = 9$	0.943118	0.900000	0.391822

Using the boundary condition Eq. 6a in the Eq. 16 we obtain,

$$a = \frac{[5K + 12(\alpha + 1)] - \sqrt{[5K + 12(\alpha + 1)]^2 - 240\alpha K}}{20\alpha} \tag{19}$$

The numerical value of a for various values of α and K is given in Tables 1 and 2. For the second iteration,

$$s_2(x) = s_1(x) + \int_0^x (\xi - x)\left[\frac{\partial^2 s_1(\xi)}{\partial \xi^2} + \alpha s_1(\xi)\frac{\partial^2 s_1(\xi)}{\partial \xi^2} - K s_1(\xi)\right] d\xi \tag{20}$$

$$s_2(x) = A + \left[\frac{2AB\alpha - KA}{2}\right]x^2 - [2C + 2AC\alpha]\,x^3 - \left[\frac{2B^2\alpha - KB - 12C}{12}\right]x^4 - \frac{3BC\alpha}{5}x^5 - \left[\frac{2BC\alpha - KC}{30}\right]x^6 - \frac{6C^2\alpha}{21}x^7 \tag{21}$$

where the values of A, B and C are given in the Eq. 18a to 18b. The first-order approximate solutions (Eq. 17) is a simple result with high accuracy. Of course the accuracy can be improved if higher-order approximate solutions required. Among the approximations (Eqs. 17 and 21), the Eq. 16 is found to be the simplest one. Hence

$$s(x) \approx s_1(x) \tag{22}$$

Equation 22 is the new simple and closed analytical expression of the substrate concentration for all values of α and K. Since $s(x) + p(x) = 1$, we can also obtain the concentration of the product from the concentration of the substrate.Using the Lyons [23] approximation (making the non-linear term $\frac{s}{1+\alpha s} \approx$ the linear term $\frac{(\alpha+s)}{(1+\alpha)^2}$). Lyons et al. [23] obtained the concentration of the substrate

$$s(x) = (1 + \alpha)\cosh\left(\frac{\sqrt{K}x}{1 + \alpha}\right)\text{sec h}\left(\frac{\sqrt{K}}{1 + \alpha}\right) - \alpha \tag{23}$$

This approximation (Eq. 22) will be valid only for certain values of α and K (Refer Fig. 2). Also this approximation will be a poor one for large K values [23]. The normalized current is given by

Table 2 The numerical value of a (for Eq. 24) for various value of α and K

	$\alpha = 0$	$\alpha = 10$	$\alpha = 20$	$\alpha = 30$	$\alpha = 40$	$\alpha = 50$	$\alpha = 60$	$\alpha = 70$	$\alpha = 80$	$\alpha = 90$
$K = 0.1$	0.047956	0.004544	0.002381	0.001613	0.001219	0.000980	0.000820	0.000704	0.000617	0.000549
$K = 1$	0.352765	0.045293	0.023787	0.016122	0.012192	0.009802	0.008196	0.007042	0.006172	0.005494
$K = 10$	0.967706	0.432402	0.235361	0.160491	0.121617	0.097869	0.081869	0.070361	0.061687	0.054916

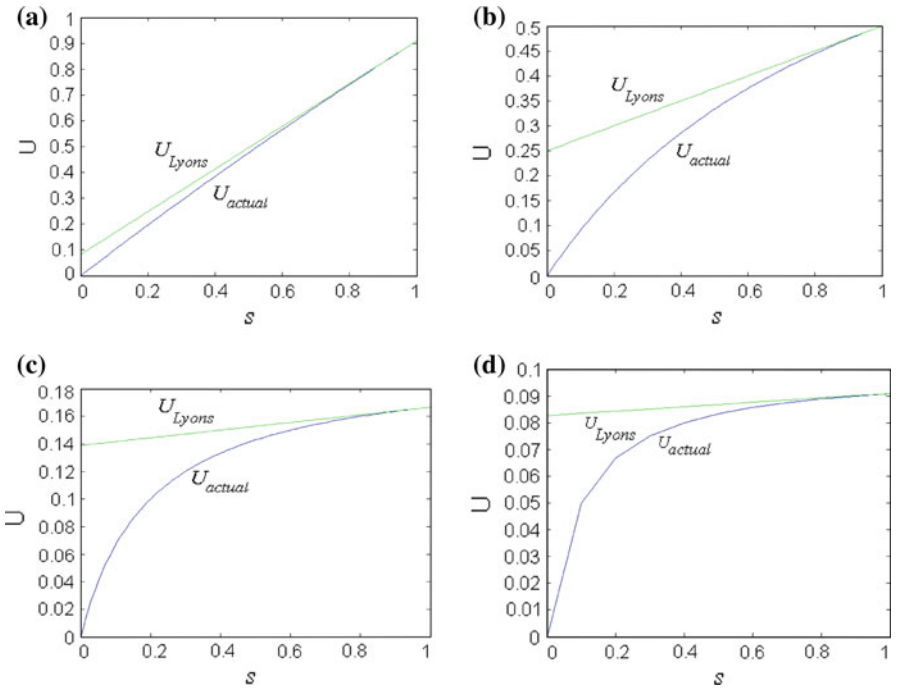

Fig. 2 Comparison of $U_{actal} = \frac{s}{1+\alpha s}$ with magic approximation $U_{Lyons} = \frac{\alpha+s}{(1+\alpha)^2}$ when **a** $\alpha = 0.1$, **b** $\alpha = 1$, **c** $\alpha = 5$, **d** $\alpha = 10$

$$y = \alpha \left(\frac{ds}{dx}\right)_{x=1} = \alpha\,(2B - 4C) = \frac{2a\alpha(2a\alpha - K - 3\alpha) + 3K\alpha}{3} \tag{24}$$

Using the approximation (Eq. 22) Lyons et al. [23] obtained the current

$$y = \alpha \left(\frac{ds}{dx}\right)_{x=1} = \alpha\sqrt{K}\tanh\left(\frac{\sqrt{K}}{(1+\alpha)}\right) \tag{25}$$

Equations 17 and 24 represents the new analytical expressions of substrate concentration and current of potentiometric biosensors exhibiting Michaelis-Menten kinetics.

5 Problem resolution including substrate concentration polarization in solution

In this case, the substrate diffusion in the solution adjacent to the polymer film is included. The transport and kinetics are described in the dimensionless form [23]

$$\frac{\partial^2 u}{\partial x^2} - \frac{Ku}{1+\alpha u} = 0 \tag{26}$$

and now the boundary conditions are [23],

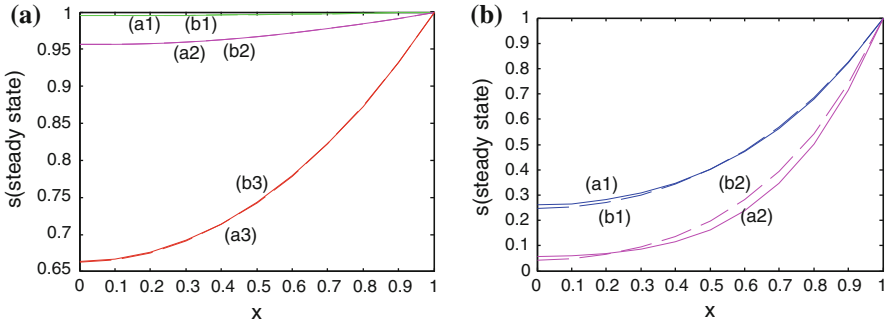

Fig. 3 Dimensionless steady state substrate concentration s(x) when $\alpha = 0.1$. The curves (a1), ($a2$) and (a3) are calculated using Eq. 16 and (b1), (b2) and (b3) are calculated using Eq. 23. **a** $K = 0.01,\ K = 0.1$ and $K = 1$, **b** $K = 4,\ K = 9$

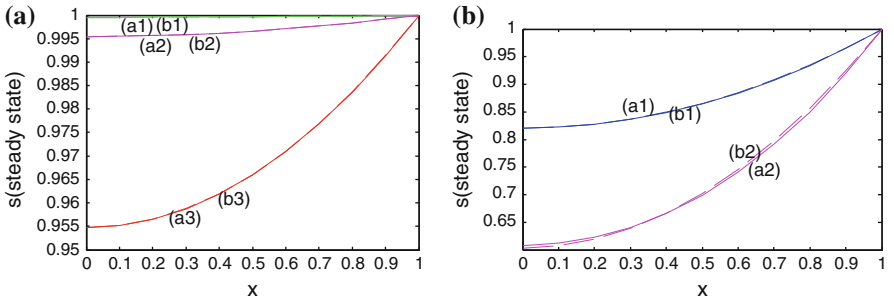

Fig. 4 Dimensionless steady state concentration of substrate s(x) when $\alpha = 10$. The curves ($a1$), ($a2$) and (a3) are calculated using Eq. 16 and (b1), (b2) and (b3) are calculated using Eq. 23. **a** $K = 0.01$ and $K = 0.1$ and $K = 1$, **b** $K = 4,\ K = 9$

$$\left.\frac{du}{dx}\right|_{x=0} = 0 \tag{27}$$

$$u(1) = u_s \tag{28}$$

$$\left.\frac{du}{dx}\right|_{x=1} = v(1-u_s) \tag{29}$$

where the Biot number v is defined as follows

$$v = \frac{k_D L}{\kappa D_F} \tag{30}$$

The non-linear Eq. 26 is solved for the above boundary conditions using variational iteration method. Using the boundary conditions (27–29), now we begin with initial guess satisfying the above boundary conditions

$$u_0 = u_s + \frac{v(1-u_s)}{2}\left[x^2 - 1\right] \tag{31}$$

where

$$u_s = \frac{-(v-\alpha v+k) \pm \sqrt{(v-\alpha v+k)^2+4\alpha v^2}}{2\alpha v} \tag{32}$$

Using the above variational iteration method we get the following approximation

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\left[\frac{\partial^2 u_n(\xi)}{\partial \xi^2} + \alpha u_n(\xi)\frac{\partial^2 u_n(\xi)}{\partial \xi^2} - K u_n(\xi)\right] d\xi \tag{33}$$

Substituting the values of u_0 from the Eq. 31 in the above Eq. 33, we get the steady state concentration (Refer Tables 4 and 5)

$$u(x) = u_1 = u_s - \frac{a}{2} + \left[K u_s - a\left(\alpha u_s + \frac{K}{2}\right) + \frac{\alpha a^2}{2}\right]\frac{x^2}{2} - \left[\alpha a^2 - Ka\right]\frac{x^4}{24} \tag{34}$$

where $a = v(1-u_s)$. The steady-state current response y is given by (refer Tables 6 and 7)

$$y = \alpha v(1-u_s) = \alpha v\left[1 - \left(\frac{-(v-\alpha v+k) \pm \sqrt{(v-\alpha v+k)^2+4\alpha v^2}}{2\alpha v}\right)\right] \tag{35}$$

Equation 34 and 35 represents the new analytical approximation of the concentration and current. Lyons et al. [23] obtained the concentration of the substrate

$$u(x) = u_s \cosh\left(\frac{\sqrt{K}x}{1+\alpha}\right)\operatorname{sech}\left(\frac{\sqrt{K}}{1+\alpha}\right) + \alpha\left[\cosh\left(\frac{\sqrt{K}x}{1+\alpha}\right)\operatorname{sech}\left(\frac{\sqrt{K}}{1+\alpha}\right) - 1\right] \tag{36}$$

Lyons et al. [23] obtained the current

$$y = \alpha v\left[1 - \frac{v - \frac{\sqrt{K}\alpha}{1+\alpha}\tanh\left(\frac{\sqrt{K}}{1+\alpha}\right)}{v + \frac{\sqrt{K}}{1+\alpha}\tanh\left(\frac{\sqrt{K}}{1+\alpha}\right)}\right] \tag{37}$$

The numerical values of u_s for various values of K when $\alpha = 0.1$ is given in Table 8.

6 Discussion

The primary result of this work is the first accurate calculation of steady state concentration of substrate (or product) and current for all values of α and K for potentiometric response of polymer modified electrode system. The lyons approximation

Table 3 Comparison of steady state current $y(\alpha, K)$ for various values of α and K

α	$K = 0.1$			$K = 1$			$K = 10$		
	Eq. 24	Eq. 25	% deviation of Eq. 24	Eq. 24	Eq. 25	% deviation of Eq. 24	Eq. 24	Eq. 25	% deviation of Eq. 24
0	0.000096	0.000096	0.000	0.000764	0.000761	0.000	0.003548	0.003151	0.000
10	0.090948	0.090885	−0.069	0.912975	0.906603	−0.703	9.622422	8.848605	−8.745
20	0.095267	0.095231	−0.038	0.955315	0.951664	−0.384	9.873839	9.452499	−4.457
30	0.096796	0.096771	−0.026	0.969953	0.967407	−0.263	9.926805	9.644004	−2.932
40	0.097578	0.097559	−0.020	0.977368	0.975417	−0.200	9.948786	9.736805	−2.177
50	0.098054	0.098038	−0.016	0.981849	0.980267	−0.161	9.960696	9.791381	−1.729
60	0.098373	0.09836	−0.013	0.984849	0.983519	−0.135	9.96814	9.827267	−1.433
70	0.098602	0.098591	−0.012	0.986998	0.985851	−0.116	9.973224	9.852643	−1.224
80	0.098775	0.098765	−0.010	0.988613	0.987604	−0.102	9.976914	9.87153	−1.068
90	0.09891	0.098901	−0.009	0.989871	0.988971	−0.091	9.979713	9.886132	−0.947
	Average deviation		−0.021	Average deviation		−0.216	Average deviation		−2.471

Table 4 Comparison of steady state concentration $u(\chi)$ for $\alpha = 0.1$ and $K = 0.01, 0.1, 1$ when $v = 0.1$

x	$v = 0.1$								
	$K = 0.01$			$K = 0.1$			$K = 1$		
	Eq. 34	Eq. 36	% deviation of Eq. 34	Eq. 34	Eq. 36	% deviation of Eq. 34	Eq. 34	Eq. 36	% deviation of Eq. 34
0	0.911884	0.912059	−0.01919	0.488117	0.485266	0.584081	0.04625	0.00098	97.88108
0.1	0.911926	0.912101	−0.01919	0.488349	0.485508	0.581756	0.046479	0.001397	96.99434
0.2	0.912051	0.912226	−0.01919	0.489045	0.486234	0.574794	0.047172	0.002653	94.3759
0.3	0.91226	0.912435	−0.01918	0.490208	0.487444	0.563842	0.048342	0.004759	90.15556
0.4	0.912552	0.912728	−0.01929	0.491836	0.48914	0.54815	0.050012	0.00773	84.54371
0.5	0.912928	0.913105	−0.01939	0.493933	0.491323	0.528412	0.052213	0.011592	77.79863
0.6	0.913388	0.913565	−0.01938	0.496499	0.493994	0.504533	0.054985	0.016378	70.21369
0.7	0.913931	0.914109	−0.01948	0.499539	0.497157	0.47684	0.058378	0.022125	62.10045
0.8	0.914558	0.914737	−0.01957	0.503054	0.500813	0.445479	0.062451	0.028883	53.75094
0.9	0.915269	0.915448	−0.01956	0.507048	0.504965	0.410809	0.067271	0.036707	45.43414
1	0.916064	0.916244	−0.01965	0.511526	0.509618	0.373002	0.072915	0.045661	37.37777
	Average deviation		−0.01937	Average deviation		0.508336	Average deviation		73.69329

Table 5 Comparison of steady state concentration $u(\chi)$ for $\alpha = 0.1$ and $K = 0.01, 0.1, 1$ when $v = 10$

x	$v = 10$								
	$K = 0.01$			$K = 0.1$			$K = 1$		
	Eq. 34	Eq. 36	% deviation of Eq. 34	Eq. 34	Eq. 36	% deviation of Eq. 34	Eq. 34	Eq. 36	% deviation of Eq. 34
0	0.994551	0.994568	−0.00171	0.945903	0.947632	−0.18279	0.496479	0.615687	−24.0107
0.1	0.994596	0.994613	−0.00171	0.946333	0.948065	−0.18302	0.498756	0.618646	−24.0378
0.2	0.994731	0.994749	−0.00181	0.947624	0.949365	−0.18372	0.505627	0.627549	−24.113
0.3	0.994957	0.994975	−0.00181	0.949778	0.951531	−0.18457	0.517205	0.642468	−24.2192
0.4	0.995274	0.995292	−0.00181	0.952796	0.954567	−0.18587	0.533684	0.663528	−24.3298
0.5	0.995681	0.995699	−0.00181	0.956682	0.958474	−0.18731	0.555333	0.690903	−24.4124
0.6	0.996179	0.996197	−0.00181	0.961438	0.963256	−0.18909	0.582497	0.724818	−24.4329
0.7	0.996767	0.996785	−0.00181	0.96707	0.968917	−0.19099	0.615599	0.765555	−24.3594
0.8	0.997445	0.997464	−0.0019	0.973582	0.975461	−0.193	0.65514	0.81345	−24.1643
0.9	0.998215	0.998234	−0.0019	0.980982	0.982894	−0.19491	0.701695	0.8689	−23.8287
1	0.999075	0.999094	−0.0019	0.989275	0.991222	−0.19681	0.755919	0.932362	−23.3415
	Average deviation		−0.00182	Average deviation		−0.18837	Average deviation		−24.1136

Table 6 Comparison of steady state current $y(\alpha, K)$ for various values of α and K

α	$K = 0.1$								
	$v = 0.1$			$v = 1$			$v = 50$		
	Eq. 35	Eq. 37	% deviation of Eq. 35	Eq. 35	Eq. 37	% deviation of Eq. 35	Eq. 35	Eq. 37	% deviation of Eq. 35
0	0	0	0	0	0	0	0	0	0
10	0.090098	0.090139	−0.04551	0.090833	0.090809	0.026422	0.090908	0.090883	0.0275
20	0.095012	0.095015	−0.00316	0.095216	0.095209	0.007352	0.095238	0.09523	0.0084
30	0.09667	0.09667	0	0.096764	0.096761	0.0031	0.096774	0.096771	0.0031
40	0.097502	0.097501	0.001026	0.097555	0.097553	0.00205	0.097561	0.097559	0.00205
50	0.098001	0.098	0.00102	0.098035	0.098034	0.00102	0.098039	0.098038	0.00102
60	0.098334	0.098333	0.001017	0.098358	0.098357	0.001017	0.098361	0.09836	0.001017
	Average deviation		−0.006	Average deviation		0.006	Average deviation		0.006

Table 7 Comparison of steady state current $y(\alpha, K)$ for various values of α and K

α	$K = 10$								
	$v = 0.1$			$v = 1$			$v = 50$		
	Eq. 35	Eq. 37	% deviation of Eq. 35	Eq. 35	Eq. 37	% deviation of Eq. 35	Eq. 35	Eq. 37	% deviation of Eq. 35
0	0	0	0	0	0	0	0	0	0
10	0.989024	4.903816	−395.824	7.298438	8.189697	−12.2116	9.075656	8.834269	2.65972
20	1.975383	7.715932	−290.604	9.155711	9.244416	−0.96885	9.519453	9.448217	0.74832
30	2.957995	8.737883	−195.399	9.534144	9.545011	−0.11398	9.675393	9.641992	0.345216
40	3.935116	9.191115	−133.567	9.680708	9.679331	0.014224	9.754931	9.735642	0.197736
50	4.903776	9.429314	−92.2868	9.757531	9.753924	0.036966	9.803165	9.790625	0.127918
60	5.858539	9.570297	−63.3564	9.80467	9.800948	0.037962	9.835535	9.826736	0.089461
	Average deviation		−167.290	Average deviation		−1.886	Average deviation		0.595

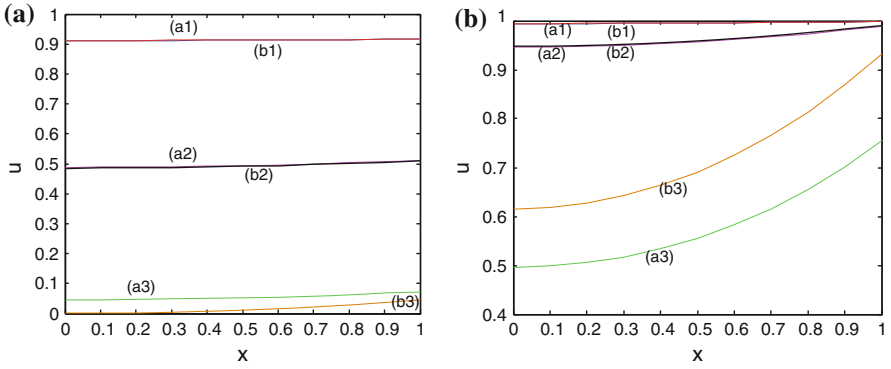

Fig. 5 Dimensionless steady state substrate concentration $u(x)$ when $\alpha = 0.1$. The curves (a1), ($a2$) and (a3) are calculated using Eq. 34 and (b1), (b2) and (b3) are calculated using Eq. 36 for $K = 0.01$, $K = 0.1$ and $K = 1$ **a** $v = 0.1$, **b** $v = 10$

Table 8 The numerical value of u_s for $\alpha = 0.1$ and various value of K

K	$v = 0.1$	$v = 10$
0.01	0.916244	0.999094
0.1	0.509618	0.991222
1	0.045661	0.932362
4	−0.03972	0.83818
9	−0.06077	0.765866

is equal to actual expression when $\alpha < 0.1$ (Refer Fig. 2a). The difference between actual expression and magic approximation is significant when $\alpha > 1$ and $s < 0.8$ (Refer Fig. 2b–d). Figures 3(a–d) and 4(a–d) shows the dimensionless steady state concentration of substrate $s(x)$ using Eq. 16 for all values of α and K. The value of these two expressions is mostly equal. Table 3 indicates the numerical value of dimensionless steady state current $y(\alpha, K)$ for all values of α and K. In this tables, our normalized current expression (Eq. 24) is compared with Lyons current (Eq. 25). Computed current values using Eq. 24 yield an average relative error of -0.021% when $K = 0.1$, -0.216% when $K = 1$ and 2.471% when $K = 10$ for all values of α when compared with Lyons current expression (Eq. 25). Tables 4 and 5 represents the dimensionless concentration profile of substrate $u(x)$ for all various values of K and v when $\alpha = 0.1$. Tables 6 and 7 represents the current for various values of K and v. In Tables 4 and 5, the concentration obtained by variational iteration method (Eq. 34) is compared with Lyons [23] expression (Eq. 36). Figure 5 shows the dimensionless concentration for various values of K and v when $\alpha = 0.1$. The computed concentration values using Eq. 34 yield an average relative error of 0.02% when $K = 0.01$, 0.51% when $K = 0.1$ and 73.69% when $K = 1$ for $v = 0.1$ and 0.002% when $K = 0.01$, 0.19% when $K = 0.1$ and 24.11% when $K = 1$ for $v = 10$ (Tables 4 and 5). Similarly, the computed current values using Eq. 35 yield an average relative error of 0.006% when $v = 0.1$, 0.006% when $v = 10$, 0.006% when $v = 50$ for

$K = 0.1$ and 167.290% when $v = 0.1$, 1.886% when $v = 10$, 0.595% when $v = 50$ for $K = 10$ (Tables 6 and 7).

7 Conclusion

The steady state potentiometric response for a polymer-modified electrode system which exhibits Michaelis-Menten kinetics has been discussed. Approximate analytical solution to the non-linear reaction-diffusion equation has been presented using variational method. A simple, straight forward and a new method of estimating the concentration of substrate or product and the corresponding current for all values of α and K has been suggested. The solution procedure can be easily extended to all kinds of non-linear equations with various complex boundary conditions in enzyme-catalyzed reaction diffusion processes.

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References

1. R. Goldman, O. Kedem, E. Katchalski, Biochem **7**, 4518 (1968)
2. P.V. Sundaram, A. Tweedale, K.J. Laidler, Can. J. Chem. **48**, 1498 (1970)
3. V. Kasche, H. Lundquist, R. Bergman, R. Axen, Biochem. Biophys. Res. Commun. **45**, 615 (1971)
4. W.J. Blaedel, T.R. Kissel, R.C. Boguslaski, Anal. Chem. **44**(12), 2030 (1972)
5. D.A. Gough, J.Y. Lucisano, P.H.S. Tse, Am. Chem. Soc. **14**, 351 (1985)
6. G. Jobst, I. Moser, G. Urban, Biosens. Bioelectron. **11**(1/2), 111 (1996)
7. S. Bacha, A. Bergel, M. Comtat, Anal. Chem. **67**, 1669 (1995)
8. S. Bacha, M. Montagne, A. Berget, AICHE J. **42**, 2967 (1996)
9. R. Baronas, F. Ivanauskas, J. Kulys, M. Sapagovas, Nonlinear Anal. Model. Control **7**, 93 (2002)
10. R. Baronas, F. Ivanauskas, J. Kulys, M. Sapagovas, Nonlinear Anal. Model. Control **8**, 3 (2003)
11. R. Baronas, F. Ivanauskas, J. Kulys, M. Sapagovas, Sensors **3**, 248 (2003)
12. D. Britz, *Digital Simulation in Eletrochemistry*, 2nd edn. (Springer, Berlin, 1988)
13. R. Baronas, F. Ivanauskas, J. Kulys, J. Math. Chem. **32**, 225 (2002)
14. P.N. Bartlett, K.F.E. Pratt, J. Electroanal. Chem. **397**, 61 (1995)
15. P.W. Carr, Anal. Chem. **49**, 799 (1977)
16. J.E. Brady, P.W. Carr, Anal. Chem. **52**, 977 (1980)
17. C. Tran-Minh, G. Broun, Anal. Chem. **47**, 1359 (1975)
18. G. Rahamathunissa, L. Rajendran, J. Math. Chem. **44**, 849 (2008)
19. M.E.G. Lyons, T. Bannon, G. Hinds, S. Rebouillat, Analyst **123**, 1947 (1998)
20. J.H. He, Comput. Methods Appl. Mech. Eng. **167**(1–2), 57 (1998)
21. J.H. He, Comm. Nonlinear Sci. Numer. Simul. **2**(4), 230 (1997)
22. S. Momani, S. Abuasad, Z. Odibat, Appl. Math. Comput. **183**, 1351 (2006)
23. M.E.G. Lyons, J.C. Greer, C.A. Fitzgerald, T. Bannon, P.N. Bartlett, Analyst **121**, 715 (1996)
24. R. Baronas, F. Ivanauskas, J. Kulys, M. Sapagovas, J. Math. Chem **34**, 227 (2003)
25. C. Phanthong, M. Somasundrum, J. Electroanal. Chem. **558**, 1 (2003)
26. S.J. Lioa, Y. Tan, A general approach to obtain series solution of non-linear differential equations. Stud. Appl. Math. **119**(4), 297–354 (2007)